AD 700 291

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January 1970

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Mimeograph Series No. 218

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* This investigation was supported in part by research grant GM-10525 from NIH, Public Health Service, at University of California, Berkeley.

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A Characterization Based on the Absolute Difference of Two I.I.D. Random Variables

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1. INTRODUCTION. Let X_1 and X_2 be two independent and identically distributed (i.i.d.) random variables whose common distribution is same as that of a non-negative random variable X. The problem considered here is to characterize all possible distributions of X which satisfy the following property H:

(1) H: The distributions of $|X_1-X_2|$ and X are identical.

For instance, it is easy to verify that the discrete distribution with $P(X = 0) = P(X = a) = \frac{1}{2}$ for some positive constant a, and the exponential distribution with probability density function (p.d.f.) f where $f(x) = \theta \exp(-\theta x)$, for $x \geq 0$, and $f(x) = 0$ elsewhere, with $\theta > 0$, both satisfy the property H. The reader may find a different characterization based on $|X_1-X_2|$ in Puri [2]. Let F denote the distribution function (D.F.) of X. It can be easily shown that if X satisfies H, the distribution of X can either be only discrete or absolutely continuous or singular and no mixture is possible. Thus one needs to consider these three possibilities separately. For the case when X is discrete let A

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denote the set of possible discrete nonnegative values that X takes. More specifically, let

$$p_y = P(X=y), \quad y \in A; \text{ with } \sum_{y \in A} p_y = 1.$$

It is clear that if there exists a $y \geq 0$ with $p_y > 0$, then in particular A contains zero with $p_0 > 0$. Furthermore, from the property H, the following relations follow easily.

$$p_0 = \sum_{x \geq 0} p_x^2 \tag{2}$$

$$p_y = 2 \sum_{x \geq 0} p_x p_{x+y}; \; y > 0. \tag{3}$$

Similar relations are satisfied by the p.d.f. f if X satisfying H is absolutely continuous.

In section 2, we show that under H, X has a moment generating function (m.g.f.) and hence all its moments are finite. Also in theorem 1, we consider the case where X is bounded. Section 3 deals with the discrete case, and theorem 2 characterizes lattice distributions satisfying H. In section 4, we consider the absolutely continuous case. Here we study a more general question; namely if X_1 and X_2 are two nonnegative independent but not necessarily identically distributed random variables (r.v.) and moreover if the distributions of X_1 and $|X_1-X_2|$ are identical, then given the distribution of X_2, what can be said about the distribution of X_1? The paper ends with a discussion in section 5, where we have few words to say about the singular case.

2. PRELIMINARY RESULTS.

In the following lemma it is shown that for an X satisfying H, its m.g.f. and hence all its moments exist.

Lemma 1. *The m.g.f. of a nonnegative r.v. X satisfying H exists.*

Proof. If X satisfying H is degenerate, it is clear that $P(X=0) = 1$, and the lemma holds trivially. Let X be nondegenerate. Then there is a number $u > 0$ such that $P(X \leq u) > \frac{1}{2}$. Using this and the property H, it follows that for every $v \geq 0$.

$$P(X > v) = P(|X_1 - X_2| > v) \geq 2\, P(X > u+v)\, P(X \leq u), \tag{4}$$

so that

$$P(X > u+v) \leq P(X > v) / 2\, P(X \leq u),$$

for all $v \geq 0$. A repeated application of this leads to

$$P(X > nu) \leq \left[\frac{1}{2\, P(X \leq u)}\right]^{n-1}; \text{ for } n = 1,2,\ldots .$$

From this one can easily show the existence of an $\alpha > 0$ such that $E(\exp(\xi X))$ exists for all $|\xi| \leq \alpha$.

The following theorem provides the answer to our problem when X is bounded.

Theorem 1. Let X be nonnegative and nondegenerate. Then the following three statements are equivalent.

(i) X is bounded and satisfies H.

(ii) X satisfies H and $P(X=0) = \frac{1}{2}$.

(iii) $P(X=0) = P(X=a) = \frac{1}{2}$, for some $a > 0$.

Proof. Clearly (iii) $\Rightarrow$ (i) and (ii). All we need to prove is that (i) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iii). Let (i) hold. Since X is bounded and nondegenerate, there exist a least upper bound $B > 0$, such that $P(X > B) = 0$ and for every $0 < \epsilon \leq B$, $P(X > B-\epsilon) > 0$. On the other hand since X satisfies H, we have for every such ϵ

$$0 < P(X > B-\epsilon) = P(|X_1 - X_2| > B-\epsilon) \leq 2\, P(X < \epsilon)\, P(X > B-\epsilon), \tag{5}$$

which implies that for every $0 < \epsilon \leq B$, $P(X < \epsilon) \geq \frac{1}{2}$ and in particular letting ϵ tend to zero we have $P(X=0) \geq \frac{1}{2}$. This implies that X must be a discrete random

variable (r.v). Using the notation introduced in section 1 for such a case, we have

$\frac{1}{2} \leq p_0 = \sum\limits_{x\geq 0} p_x^2 < 1$. This yields

$$p_0(1-p_0) = \sum_{x>0} p_x^2 \leq \left(\max_{x>0} p_x\right)\left(\sum_{x>0} p_x\right) = \left(\max_{x>0} p_x\right)(1-p_0),$$

so that $p_0 \leq \max\limits_{x>0} p_x$. Hence we have

$$\tfrac{1}{2} \leq p_0 \leq \max_{x>0} p_x \leq 1-p_0 \leq \tfrac{1}{2},$$

so that $p_0 = \max\limits_{x>0} p_x = \frac{1}{2}$, which implies that $P(X=0) = P(X=B) = \frac{1}{2}$. This proves that (i) ⇒ (iii) with $B = a$. Now let (ii) hold. Since $p_0 = \frac{1}{2}$, X must be a discrete r.v if it has to satisfy H. For this we have already seen that $p_0(1-p_0) = \sum\limits_{x>0} p_x^2$. This means that we have

$$\sum_{x>0} p_x^2 = \tfrac{1}{4} \quad \text{and} \quad \sum_{x>0} p_x = \tfrac{1}{2}.$$

But this holds if and only if $p_x = \frac{1}{2}$ for some $x = a > 0$, so that (iii) holds.

Q.E.D.

Before closing this section we wish to remark that for the nondegenerate discrete case, for X satisfying H we must have $0 < p_0 \leq \frac{1}{2}$. That $p_0 > 0$ follows from (2) and the fact that $\sum\limits_{x\geq 0} p_x = 1$. That $p_0 \leq \frac{1}{2}$ follows from the fact that for every $y > 0$ with $p_y > 0$, $p_y \geq 2\, p_0 p_y$ under H.

3. DISCRETE CASE. We now consider the case where X is discrete and satisfies the following additional condition C.

(6) C: There exists an interval $(\delta_1, \delta_2]$ with $0 \leq \delta_1 < \delta_2 < \infty$ such that $P(\delta_1 < X \leq \delta_2) = 0$.

Using the notation of section 1, we first prove three lemmas needed to prove the main result of Theorem 2.

Lemma 2. Let X be discrete, nondegenerate and satisfy H and the condition C. Then

(i) $\tau = \inf\{x: \ x > 0,\ p_x > 0\} > 0$ and $p_\tau > 0$, and

(ii) the set of possible values of X is given by $k\tau$; $k = 0,1,2,\ldots$.

Proof. (i) Since X is nondegenerate, it is clear that the set $\{x: x > 0,\ p_x > 0\}$ is nonemty. Again if $p_0 = \frac{1}{2}$, (i) and (ii) are satisfied in view of Theorem 1, so that let $0 < p_0 < \frac{1}{2}$. By Theorem 1, this means that X is not bounded. Let S be the set of possible values of X. In view of the property H, it is easy to show that S forms a positive linear space in integers. By this we mean that if $x_i \in S$, $i = 1,2,\ldots$, then $|\sum_i n_i x_i| \in S$ for all integer values (positive or negative) of n_i's. Now for an unbounded set with this property, it is not difficult to show that either this set is dense everywhere over $[0,\infty)$ or is a lattice. On the other hand in view of condition C, it cannot be dense everywhere. Hence the lemma follows. Q.E.D.

In view of lemma 2, let p_k denote the probability $p(X=k\tau)$, for $k = 0,1,2,\ldots$, so that $\sum_{k=0}^{\infty} p_k = 1$. The analogues of (2) and (3) are given by

$$(9) \qquad p_0 = \sum_{i=0}^{\infty} p_i^2$$

$$(10) \qquad p_k = 2 \sum_{i=0}^{\infty} p_i\, p_{i+k};\ k = 1,2,\ldots$$

We shall now restrict to the case with $0 < p_0 < \frac{1}{2}$. Thus the set of possible values of X, in view of Theorem 1, must be infinitely denumerable. Furthermore, from (12) of the following lemma it follows that under condition C, $p_k > 0$, for $k = 1,2,\ldots$.

Lemma 3. *Let* H *and* C *hold and also let* $0 < p_0 < \frac{1}{2}$. *Then for* $k = 1,2,\ldots$,

(11) $$p_k \geq \left(\frac{2\, p_1}{1-2\, p_0} \right) \cdot p_{k+1}$$

and

(12) $$p_{k+1} \geq \frac{2\, p_1(1-2p_0)}{[(1-2p_0)^2 + 4\, p_1^{\,2}]} \cdot p_k$$

Proof. (11) follows easily from (10) by noticing that for $k \geq 1$,

(13) $$p_k(1-2p_0) - 2p_1 p_{k+1} = 2 \sum_{i=2}^{\infty} p_i\, p_{i+k} ,$$

and that the right side of (13) is nonnegative. To prove (12), we first notice from (10) that for $k \geq 1$,

(14) $$p_{k+1} = 2 \sum_{i=0}^{\infty} p_i\, p_{i+k+1} ,$$

or equivalently

(15) $$p_{k+1}(1-2p_0) = 2 \sum_{i=1}^{\infty} p_i\, p_{i+k+1} .$$

Then using (11) for each p_i on the right side of (15) we have

(16) $$p_{k+1}(1-2p_0) \geq \frac{4p_1}{1-2p_0} \sum_{i=1}^{\infty} p_{i+1}\, p_{i+k+1} = \frac{2p_1}{1-2p_0} \left(2 \sum_{i=2}^{\infty} p_i\, p_{i+k}\right)$$
$$= \frac{2p_1}{1-2p_0} (p_k - 2p_0 p_k - 2p_1 p_{k+1}) .$$

Here at the end of (16) we have again used (10). Finally (12) follows immediately from (16) after a little simplification. Q.E.D.

For each sequence $\{p_k\}$ satisfying H (or equivalently (9) and (10)) and with $0 < p_0 < \frac{1}{2}$, define

$$\beta = \sup \{b:\ \ b > 0 \text{ satisfying } p_k \geq b\ p_{k+1} \text{ for all } k \geq 1\}$$

and

$$\gamma = \sup \{c:\ \ 0 < c < 1, \text{ satisfying } p_{k+1} \geq c\ p_k \text{ for all } k \geq 1\},$$

so that

(17) $$p_k \geq \beta\ p_{k+1}\ ;\ k = 1,2,\dots\ ,$$

and

(18) $$p_{k+1} \geq \gamma\ p_k\ ;\ k = 1,2,\dots\ .$$

From lemma 3 it follows that for every sequence $\{p_k\}$ satisfying H and with $0 < p_0 < \frac{1}{2}$, there always exists positive β and γ. Also by definition of β, it is clear that for every such sequence $0 < \beta < p_1/p_2$. Here $p_2 > 0$; in fact because of (12) $p_k > 0$ for all $k \geq 1$. Also γ has to be strictly between 0 and 1. That it cannot be equal to one follows from (18) and the fact that $\sum\limits_{i=0}^{\infty} p_i$ converges.

The following lemma is the essential lead to the main theorem of this section.

Lemma 4. <u>For every sequence</u> $\{p_k\}$ <u>with</u> $0 < p_0 < \frac{1}{2}$ <u>and satisfying</u> H, $\beta\ \gamma = 1$.

Proof. From (17) and (18) it is clear that $\beta\ \gamma \leq 1$. It is sufficient then to prove that $\beta\ \gamma \geq 1$. From (14) we have for $k \geq 2$,

(19) $$p_k\ (1-2p_0) = 2 \sum_{k=1}^{\infty} p_i\ p_{i+k}\ .$$

Using (18) on the right side of (19) for each p_i we have

(20) $$p_k(1-2p_0) \leq \frac{2}{\gamma} \sum_{i=1}^{\infty} p_{i+1}p_{i+k} = \frac{2}{\gamma} \sum_{i=2}^{\infty} p_i\ p_{i+k-1}$$

$$= \frac{1}{\gamma} [p_{k-1} - 2p_0p_{k-1} - 2p_1p_k]\ ,$$

which after simplification, yields for $k = 2,3,\ldots$;

$$p_{k-1} \geq [\gamma + \frac{2p_1}{1-2p_0}]\, p_k \; ,$$

or equivalently for $k = 1,2,\ldots$,

$$p_k \geq [\gamma + \frac{2p_1}{1-2p_0}]\, p_{k+1} \; . \tag{21}$$

Comparing (17) and (21) and keeping the definition of β in mind, we have

$$(\beta - \gamma) \geq \frac{2p_1}{1-2p_0} \; . \tag{22}$$

Again, using (17) on the right side of (19), we have for $k \geq 2$,

$$\begin{aligned} p_k(1-2p_0) &\geq 2\beta \sum_{i=1}^{\infty} p_{i+1}\, p_{i+k} = \beta(2 \sum_{i=2}^{\infty} p_i\, p_{i+k-1}) \\ &= \beta\, [p_{k-1} - 2p_0 p_{k-1} - 2p_1 p_k] \; . \end{aligned} \tag{23}$$

On simplification, (23) yields for $k \geq 2$,

$$p_k \geq \frac{\beta(1-2p_0)}{(1-2p_0 + 2p_1\beta)}\, p_{k-1} \; ,$$

or equivalently for $k \geq 1$,

$$p_{k+1} \geq \frac{\beta(1-2p_0)}{(1-2p_0 + 2p_1\beta)}\, p_k \; . \tag{24}$$

Finally comparing (18) and (24) and using the definition of γ, we obtain

$$\gamma \geq \frac{\beta(1-2p_0)}{(1-2p_0 + 2p_1\beta)}$$

or after simplification

$$(\beta-\gamma) \leq \frac{2p_1}{1-2p_0}\, \beta\, \gamma \; . \tag{25}$$

Now it easily follows from (22) and (25) that $\beta\,\gamma \geq 1$. Q.E.D.

We are now in a position to state and prove the main theorem of this section.

Theorem 2. *Let X_1 and X_2 be two independent copies of a nonnegative discrete random variable X satisfying condition C. Then X and the absolute difference $|X_1 - X_2|$ have the same distribution if and only if the distribution of X is given for some positive constant τ, by*

$$(26)\qquad \begin{cases} \Pr(X = 0) = p_0 \\ \Pr(X = k\tau) = 2p_0(1-p_0)\,(1-2p_0)^{k-1};\ k = 1,2,\ldots, \end{cases}$$

where either $p_0 = 1$ or $0 < p_0 \le \frac{1}{2}$.

Proof. The case with $p_0 = 1$ is that of a degenerate r.v. X. Also we have argued before that for a nondegenerate X, we must have $0 < p_0 \le \frac{1}{2}$. The case with $p_0 = \frac{1}{2}$ is covered in theorem 1. Let us assume then that $0 < p_0 < \frac{1}{2}$. From lemma 4 and equations (17) and (18) it follows that

$$(27)\qquad p_{k+1} = \gamma p_k\ ;\ k = 1,2,\ldots$$

or equivalently

$$(28)\qquad p_k = \gamma^{k-1} p_1\ ;\ k = 1,2,\ldots.$$

Now it is easy to show using (9) and the fact that $\sum_{i=0}^{\infty} p_i = 1$, that $\gamma = (1-2p_0)$ and $p_1 = 2p_0(1-p_0)$. Q.E.D.

4. ABSOLUTELY CONTINUOUS CASE. Let f(x) denote the p.d.f of the nonnegative r.v. X with the property H. The property H is then equivalent to f(x) satisfying the relations

$$(29)\qquad \int_0^\infty f(x)dx = 1\ ;\ f(t) = 2\int_0^\infty f(x+t)\, f(x)dx\ ;\ \text{for all } t \ge 0.$$

Furthermore in view of theorem 1, X is unbounded. The following lemma gives certain properties of an f satisfying (29), which we shall need later.

Lemma 5. Let the p.d.f $f(x)$ of a nonnegative r.v X satisfy (29). Then it also satisfies the following:

(i) $f(x)$ is lower semicontinuous for all $x \geq 0$.

(ii) $f(x) > 0$, for all $x \geq 0$.

Proof. (i) From (29) for $t = 0$, we have

$$(30) \qquad f(0) = 2\int_0^\infty f^2(x)dx \; ; \int_0^\infty f(x)dx = 1 \; ,$$

so that we must have $f(0) > 0$. On the other hand since

$$\psi(z) = \int_0^\infty f(y+z)\, f(y)dy \; ; \; -\infty < z < \infty \; ,$$

is the p.d.f of X_1-X_2, or equivalently the p.d.f. of the convolution of X_1 and $-X_2$, $\psi(z)$ is lower semicontinuous for all $-\infty < z < \infty$. By virtue of (29) therefore, $f(t)$ is lower semicontinuous for all $t \geq 0$.

(ii) Assume that there exists an interval (a,b) with $\epsilon < a < b$, such that $f(x) = 0$ for all $x \in (a,b)$. Since $f(0) > 0$ and $f(x)$ is lower semicontinuous at zero, there is an $\epsilon > 0$, with $a < b - \epsilon/2$, such that $f(x) > 0$, for all $x \in [0,\epsilon]$. Using this and (29), it is now easy to show that $f(z) = 0$, for all $a < z < b + \epsilon/2$, a.e.μ. By an induction argument we then have $f(z) = 0$ for all $a < z < b + n\,\epsilon/2$, a.e.$\mu$, for $n = 1,2,\ldots$. Letting $n \to \infty$, we have $f(z) = 0$ for all $a < z < \infty$, a.e.μ. But this implies that X is bounded, which is a contradiction. Thus there exists no interval $I \subset [0,\infty)$ with $\mu(I) > 0$ such that $f(x) = 0$ for all $x \in I$. This implies that $f(x) > 0$ for all $x \geq 0$, a.e.μ. Now let $f(x_0) = 0$ for some $x_0 > 0$. Then

$$\int_0^\infty f(x+x_0)\, f(x)dx = 0 \Rightarrow \int_a^b f(x+x_0)\, f(x)dx = 0, \text{ for } 0 < a < b < \infty$$

$$\Rightarrow f(x+x_0) = 0, \text{ for all } a < x < b \text{ with } f(x) > 0, \text{ a.e.}\mu.$$

But $\mu\,[x\colon\ a < x < b,\ f(x) > 0] = b-a$, which also yields $\mu\,[x+x_0 : a < x < b, f(x) > 0] = b-a > 0$. This contradicts the fact that $f(x) > 0$, for all $x \geq 0$ a.e.μ. Thus $f(x) > 0$, for all $x \geq 0$. Q.E.D.

We shall now consider a more general problem. Let X and Y be two nonnegative independently but not necessarily identically distributed random variables. Let F and G denote the D.F's of X and Y respectively. Given F and that the distributions of Y and $|Y-X|$ are identical, what can we say about the distribution of Y, i.e. about G? The reader may find in Feller [1] a treatment of this problem considered for a somewhat restricted case. That the distributions of Y and $|Y-X|$ are identical is equivalent to the relation

$$G(t) = \int_0^\infty G(x+t)\, dF(x) + \int_0^\infty F(y+t)\, dG(y)\ ;\ \text{for all } t \geq 0\ . \tag{31}$$

The following theorem provides an answer to the question raised above.

Theorem 3. Let X and Y be two nonnegative independent random variables with F and G as their respective D.F.'s. Let $E\,X < \infty$ and F have an absolutely continuous part. Then G satisfies (31) if and only if G is absolutely continuous with p.d.f. g where

$$g(y) = [1-F(y)]\ /\ EX\ ;\ \text{for all } y \geq 0. \tag{32}$$

Proof. It is easy to verify that $g(y)$ of (32) does satisfy (31). All we need to show is that this is the unique g that satisfies (31). To this end, consider a sequence of i.i.d. random variables $X_1, X_2, X_3, \ldots$, with their common distribution same as that of X. Define another sequence of random variables Z_k recursively by

$$Z_1 = X_1,\ Z_{n+1} = |Z_n - X_{n+1}| \ . \tag{33}$$

Then clearly $\{Z_n\}$ is a Markov chain (M.C) with state space $[0,\infty)$ and the transition D.F. H, given by

$$dH(x|y) = dF(x-y) + dF(x+y) \ , \tag{34}$$

so that if G_n is the D.F. of Z_n, it is easily observed that $G_1 = F$ and for $n = 2,3,\ldots,$

$$G_n(y) = \int_0^\infty G_{n-1}(x+y)\, dF(x) + \int_0^\infty F(x+y)\, dG_{n-1}(x) \ . \tag{35}$$

Letting $n \to \infty$, we observe that any solution G of (31) is a stationary distribution of the M.C. $\{Z_n\}$. We have already observed that $g(y)$ given by (32) is such as stationary distribution. That this is the unique stationary distribution and hence the unique solution of (31) follows from the fact that the above M.C. defined on $[0,\infty)$ is indecomposable, which is a simple consequence of the fact that F has an absolutely continuous part. Q.E.D.

In answer to our original question, we now have the following theorem.

__Theorem__ 4. __Let X_1 and X_2 be two independent copies of a nonnegative r.v. X with p.d.f. $f(x)$. Then X and $|X_1-X_2|$ have the same distribution, if and only if for some $\theta > 0$,__

$$f(x) = \begin{cases} \theta\, e^{-\theta x} \ , & \text{for } x \geq 0 \\ 0 & \text{elsewhere} \ . \end{cases} \tag{36}$$

__Proof__. Clearly if $f(x)$ satisfies (36), the distributions of X and $|X_1-X_2|$ are identical. Assuming now that the distributions of X and $|X_1-X_2|$ are the same, it is easily seen that $f(x)$ satisfies the conditions of theorem 3, in view of lemmas 1 and 5. On the other hand comparing (29) and (31), we have under H, $g(y) = f(y)$, so that replacing $g(y)$ with $f(y)$ in (32), and solving the resulting equation for $f(y) = F'(y)$ we obtain (36) with $\theta = EX$. Q.E.D.

5. A FEW CONCLUDING REMARKS. The lines of proof adopted for theorem 3 and hence of theorem 4, in principle should also work for the discrete case of section 3. Let X and Y be two appropriate nonnegative discrete r.v, both independently but not necessarily identically distributed with $\{p_x\}$ and $\{p_y\}$ as the set of their probabilities (as defined in section 1). Given that the distributions of Y and $|Y-X|$ are identical, the analogue of equation (29) is given by

$$(37)\qquad \begin{cases} q_0 = \sum\limits_{x\geq 0} p_x\, q_x \\ \\ q_y = \sum\limits_{x\geq 0} p_x\, q_{x+y} + \sum\limits_{x\geq 0} q_x\, p_{x+y}\ ; \text{ for } y > 0. \end{cases}$$

However, here essentially it is a matter of first guessing a general solution of (37) for q_y's satisfying $\sum\limits_{y\geq 0} q_y = 1$, in terms of p_x's, an analogue of (32). After this, replacing q_y's with p_x's in this solution, p_y's can be explicitly obtained to yield the answer to our original problem.

Concerning the singular case of an X with property H, at present we can only say in view of theorem 1, that X has to be unbounded. On the other hand let us consider again the approach adopted in section 4. Let F(x) and G(y) respectively be the continuous distribution functions of two nonnegative independent random variables X and Y. This will cover both absolutely continuous and singular cases of our problem. Introduce a M.C similar to the one of section 4, defined on $[0,\infty)$, but with the assumption that X has the continuous D.F. F(x), so that (31) and (35) are still satisfied. Any solution of (31) is a stationary D.F. G of our M.C. $\{Z_n\}$. On the other hand, if $EX < \infty$, it is easily verifiable that

$$(38)\qquad dG(y)/dy = [1-F\ (x)]\ /\ EX\ ;\ y \geq 0\ ,$$

is a solution of (31) and hence a stationary D.F. of M.C. $\{Z_n\}$. The only problem

here is to show that (38) is the unique solution of (31). For this we need to show that the M.C. $\{Z_n\}$ is indecomposable. Once this is established, (38) is the unique solution of (31). The solution to our problem is then obtained by replacing G with F in (38) and solving this for F. This turns out to be the same as (36). Thus, subject to the uniqueness of the solution of (31), the solution to our original problem would be (36) even when D.F. of X is given to be only continuous. This would mean that there is no singular distribution with the property H. Our conjecture is that this is in reality the case.

Finally, in section 3 for the discrete case the result of theorem 2 was proved subject to the condition C. Our conjecture is that this result holds even without this extra condition.

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Unclassified

Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Purdue University	2a. REPORT SECURITY CLASSIFICATION Unclassified
	2b. GROUP

3. REPORT TITLE
A Characterization Based on the Absolute Difference of Two I.I.D. Random Variables.

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)
Technical Report, January, 1970

5. AUTHOR(S) (Last name, first name, initial)
Puri, Prem S. and Rubin, Herman

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
January, 1970	16	2

8a. CONTRACT OR GRANT NO. N00014-67-A-0226-0008 and GM-10525	9a. ORIGINATOR'S REPORT NUMBER(S) Mimeo Series # 218
b. PROJECT NO.	
c.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
d.	

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13. ABSTRACT Let X be a nonnegative random variable with X_1 and X_2 as its two independent copies. The problem considered here is to characterize all the nonnegative distributions with the property that the distribution of the absolute difference $|X_1-X_2|$ is the same as that of X. It is shown that in general such a distribution has to be either purely discrete, or absolute continuous or singular and that it cannot be their mixture. It is shown that among lattice distributions the only distribution that enjoys the above property is given for some positive constant a, by $P(X = 0) = p_0$; $P(X = k\,a) = 2p_0(1-p_0)(1-2p_0)^{k-1}$; $k = 1,2,3,\ldots$, where either $p_0 = 1$ or $0 < p_0 \leq \frac{1}{2}$. When X is absolutely continuous, it is shown that in order to enjoy the above property, its distribution has to be exponential with p.d.f. $f(x) = \theta e^{-\theta x}$, for $x \geq 0$, and zero elsewhere, with $\theta > 0$. The case when X is singular is not completely solved. However, it is shown that in this case X cannot be bounded. A partial solution has also been given for the following more general question. Let X and Y be two nonnegative independently but not necessarily identically distributed random variables with the property that the distributions of Y and $|Y-X|$ are identical. Given the distribution of X, what can we say about the distribution of Y?

DD FORM 1 JAN 64 1473

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14 KEY WORDS	LINK A ROLE	LINK A WT	LINK B ROLE	LINK B WT	LINK C ROLE	LINK C WT
1. Characterization of distributions.						
2. Absolute difference of two nonnegative random variables.						
3. Exponential distribution.						

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